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Efficiency, egalitarianism, stability and social welfare in infinite dimensional economies

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Abstract

In the framework of general equilibrium theory, this paper considers the existing relationship between the concepts of egalitarianism, efficiency and fairness, in a pure exchange economy with infinitely many goods. We consider the possibility of achieving an efficient egalitarian allocation in a decentralized way. Finally we introduce an index to measure the degree of inequality in a given economy.

JEL Classification: D4, D6.
Keywords: Fairness, efficiency, economic welfare.

Resumen

En el marco de la teoría del equilibrio general, este trabajo examina la relación existente entre los conceptos de igualitarismo, eficiencia y justicia, en una economía de intercambio puro con bienes infinitos. En este trabajo, estamos considerando que es posible lograr una distribución igualitaria eficiente de una manera descentralizada. Por último, se introduce un índice para medir el grado de desigualdad en una economía determinada.

Clasificación JEL: D4, D6.
Palabras Clave: Justicia, eficiencia, bienestar económico.

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Introduction

In this paper we discuss the relationship between Pareto optimality (i.e., efficiency), social welfare and equality. We show that there exists an egalitarian and efficient allocation, ensuring at the same time, fairness and social egalitarianism. The studies on this topic can be divided into two major lines according to the authors approach. Our characterization of the egalitarian allocation follows the approach given by authors like d'Aspremont and Gevers (1977), Hammond (1976), and Strasnick (1976); who worked in the framework of social choice theory. A different approach is followed by authors such as Roth (1979) and Imair (1983), who have worked following a similar approach to the used by Nash (1950) in his classic paper.

We argue that, by transfers, it is possible to obtain a fair (egalitarian) and efficient economy, this is equivalent to the fact that agents in an initial time, agree with a distribution over the vector of the aggregate initial endowments. As we will see this allocations admits a natural maximin interpretation. We consider two classical and apparently different points of view. The point of view of the general equilibrium theory, followed by Arrow (1951), and on the other hand the point of view of the distributive justice, followed by Sen (1977) and Rawls (1999). These two and apparently antagonist points of view, can be summarized following the Negishi approach (Negishi, 1960).

The Negishi's approach provides an alternative approach to the Arrow-Debreu model to find the Walrasian equilibria of the economy. The rational of this approach is to characterize efficient allocations as solutions of a central planner's problem and then use welfare theorems to establish the equivalence of this set of solutions with the set of equilibrium allocations. The main advantage is that it can be extended naturally to the case of infinite dimesional economies, allowing us to work on a finite dimensional set of social weights. Precisely the Negishi's method is widely used to calculate and characterize the set of Walrasian equilibria of infinite dimensional economies by a finite set of equations, see for instance Accinelli and Plata (2011). Our main contribution in this paper is to relate, via the method of Negishi, social equality or social inequality and efficiency with competitive market opportunities. The use of such method allows us to extend the analysis to economies modeled on infinite dimensional spaces, in a natural way.

By efficiency we understand efficiency in the Pareto optimality sense. The concept of equality considered in this work is close to the idea that John Rawls has called “equality of fair opportunity” (see Rawls, 1999). Finally, stability is introduced as a concept of social stability of the economy, in the sense that the action of individuals who prefer to play in a non-cooperative way, can be blocked by the action of the rest of the society.

This work is organized as follows. In the next section we introduce the main characteristics of the economies considered. In the next section we introduce the model. Following the ussual approach we consider only a pure exchange economy. Most of the results will continue being valid if we introduce production in a classical way. However it would be technically more demanding. In section 3 we analyze the relationship between efficiency and social welfare. Next, in section 4 we introduce some considerations on the egalitarianism. In section 5 some considerations on the stability of the egalitarian solution, are gieven. In section 6 we introduce the definition of unequal economy and we argue on the possibilities to reach egalitarian allocations in a decentralized way. In this section we introduce two equivalent indices to measure inequality. Finally we give some conclusions.

1. The model

We consider an exchange economy composed by n consumers, and where the consumption set is described by the positive cone of a Riesz space E.

$$\mathcal{E} = \{X_i, u_i, w_i, i \in I\}$$

where $I = \{1,2, \ldots n\}$ is an index set symbolizing the agents of the economy. We assume that the consumption set $X_i = E^+$ is the same for every agent and it is the positive cone of a Riesz Space E. By E^* we symbolize the topological dual space of E i.e., the set of the continuous functionals in E. The utility functions are real, strictly concave, monotone, and continuous functions, defined by, $u_i: E^+ \to R, i = 1, \ldots, n$. The initial endowment of each consumer i will be denoted by $w_i > 0$ and the total endowment by $W = \sum_{i=1}^{n} w_i$.

Definition 1. An allocation $x = (x_1, \ldots, x_n)$ is an specification of a consumption bundle, where $x_i \in E^+$ for each consumer $i \in I$.

Let us define the feasible set $\mathcal{F} \subset (E_+)^n$ as the set of consumption bundles,

$$\mathcal{F} = \{x = (x_1, \ldots., x_n): x_i \in E^+; \forall i \in I, : \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} w_i = W\}, \qquad (1)$$

and the utility possibility set:

$$U = \left\{ \begin{array}{c} u \in R^n : there\ is\ a\ feasible\ allocation\ x\ such\ that \\ u_i \leq u_i(x_i),\ \forall i \in I \end{array} \right\} \tag{2}$$

Definition 2. The set of admissible directions for the point $x \in E^+$ is the direction given $v \in A_x \subset E$ where:

$$A_x = \{v \in E : x + v = y : y \in E^+\}.$$

Notice that if v is admissible for x then $x + tv \in E^+$ for all $0 \leq t \leq 1$.

Remark 1. (Notation) Given an allocation $x = (x_1, \dots, x_n)$, by $u(x) \in R^n$ we symbolize the vector $u(x) = (u_1(x_1), \dots . u_n(x_n))$.

Note that under the assumptions of this work, the utility possibility set is convex. This result follows straightforward from the concavity of the utility functions because: If $u^1, u^2 \in U$ then there exist $x^1, x^2 \in \mathcal{F}$ such that $u_i^1 \leq u_i(x_i^1)$ and $u_i^2 \leq u_i(x_i^2)$. So, $\alpha u_i^1 + (1-\alpha) u_i^2 \leq u_i(\alpha x_i^1 + (1-\alpha) x_i^2),\ \forall i \in \{1,2, \dots, n\}$. Since $\mathcal{F}$ is a convex set the affirmation holds.

Definition 3. A feasible allocation x is Pareto optimal if there is no other allocation x' such that $u_i({x'}_i) \geq u_i(x_i)$ for all $i \in I$ and $u_k({x'}_k) > u_k(x_k)$. By PO we symbolize the subset of F of the Pareto optimal allocations.

From the previous definition it follows directly that the Pareto optimal concept does not concern with fairness. It is a concept related to efficiency in the sense that an allocation is Pareto optimal if there is no waste, i.e: it is not possible to improve any consumer's utility without making someone worse off.

We assume that the exchange economy E satisfy the closedness condition, i.e., U is a closed set of R^n. Recall that under the closedness condition, the existence of a rational Pareto optimal allocations follows and the boundary of the utility possibility set correspond to the Pareto optimal allocations. Myopic utility functions is a sufficient condition for closedness condition, see for instance Aliprantis, Brown and Burkinshaw (1990).

Directly from the definition of Pareto optimality, it follows that the Pareto optimal allocations must belong to the boundary of the utility possibility set. The boundary of this set will be denoted by UP and is defined by:

$$UP = \begin{Bmatrix} u \in U \colon \nexists \ u' \in U \colon u_i' \geq u_i \ \ \forall \ \ i \in I, \ \ and \ \ \ u_k' > u_k \\ for \ \ some \ \ k \in I \end{Bmatrix} \tag{3}$$

The next lemma is straightforward:

Lemma 1. A feasible allocation x is a Pareto optimal if and only if

$(u_1(x_1), \ldots, u_n(x_n)) \in UP$.

Proof: Since utilities are monotone and strictly concave, they are strictly monotone and then a feasible allocation x can be Pareto optimal if and only if the utility vector $u = (u_1(x_1), \ldots, u_n(x_n)) \in UP$.

2. Pareto optimality and social welfare optimum

In this section we discuss the relationship between the Pareto optimality concept and the maximization of a social welfare function.

We will consider a particularly simple social welfare function given by: $U_\lambda \colon \mathcal{F} \to R$, and defined as:

$$U_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i) \tag{4}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is fixed and can be considered as a vector of social weights. Since the social welfare function should be nondecreasing in the individual utility, we can consider $\lambda \geq 0$. Moreover we can assume that λ belongs to the $n-1$ dimensional simplex Δ^{n-1}. This function summarizes the social welfare associated to the allocation x, but certainly this social value changes if λ changes.

Note that if the utility vector $u = (u_1, \ldots, u_n)$ is associated with a Pareto optimal allocation $x \in \mathcal{F}$, being $u_i = u_i(x_i)$ for each $i \in \{1, \ldots, n\}$ then, u is in the boundary of the possibility utility set. This observation suggests the next proposition:

Proposition 1. The set of Pareto optimal allocations is homeomorphic to the simplex Δ^{n-1}.

This is a consequence of the following lemma.

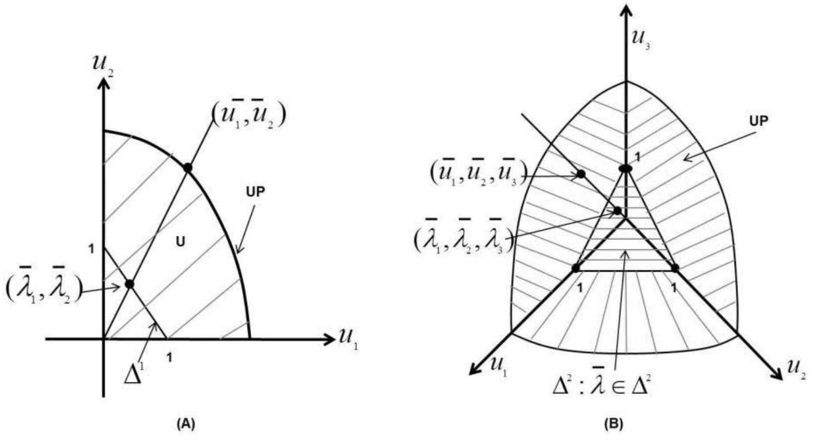
Lemma 2. If utilities $u_i, i = 1, \ldots, n$ are strictly concave, then UP is homeomorphic to the $n - 1$ simplex.

Proof: Consider the function $\xi : UP \to \Delta$ defined by $\xi(u) = \frac{1}{u_1 + \ldots + u_n} u$. Since ξ is a homeomorphism the result follows.

This homeomorphism is shown in figure (1) (A) for two consumers, and (B) for the case of three consumers.

Figure 1

The homeomorphism between Δ and UP for $n = 2$ and $n = 3$



The proposition (1) is a straightforward conclusion from this lemma.

Proof of the proposition: Let us symbolize by $\mathcal{PO}$ the set of Pareto optimal allocations, so for each $u \in UP$ there exists $x \in \mathcal{PO}$ such that $u = u(x)$ and reciprocally. Consider $\phi : \mathcal{PO} \to UP$ given by $\phi(x) = u$ and $\psi : \mathcal{PO} \to \Delta^{n-1}$ given by $\psi(x) = \xi(\phi(x)) = \lambda$.

If our interest is to find an allocation maximizing the social welfare, it is clear that this allocation must be chosen from Pareto optimal allocations. Suppose that for a fixed $\lambda \in \Delta^{n-1}$, we consider the social utility function $U_\lambda(x)$, so it makes sense to select an allocations in $\mathcal{F}$ maximizing this function, i.e, solving the following maximization problem:

$$\max_{x \in \mathcal{F}} U_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i) \tag{5}$$

For a fixed $\bar{\lambda} \in \Delta_+^{n-1}$, it is straightforward to see that a necessary condition to be $x \in \mathcal{F}$ a solution of the maximization problem (5), is that $x \in \mathcal{PO}$, but certainly it is not a sufficient condition.

Let $u = (u_1, \dots, u_n) = (u_1(x_1), \dots, u_n(x_n))$ be a utility vector, so the next equalities hold:

$$U_{\bar{\lambda}}(x) = \sum_{i=1}^{n} \bar{\lambda}_i u_i = \bar{\lambda} u.$$

This number can be considered as the social value of the allocation x.

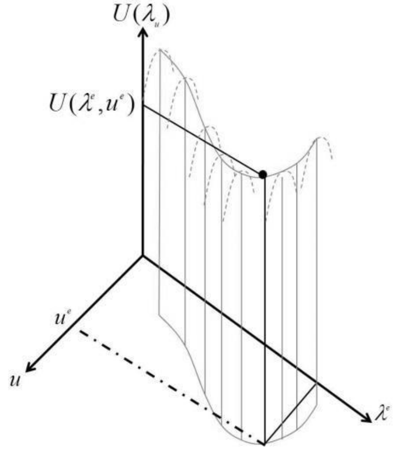
Thus the problem (5) can be written as:

$$\max_{u \in UP} \bar{\lambda} u$$

and the solution of this problem corresponds to an utility vector $\bar{u} \in UP$ such that $\bar{\lambda}\bar{u} \geq \bar{\lambda} u \ \ \forall u \in U$.

Figure 2

The Pareto optimal allocations and the equalitarian allocation



These considerations are summarized in the next proposition:

Proposition 2. For each $\lambda \in \Delta^{n-1}$ the social welfare function $U_\lambda: F \to R$ defined by $U_\lambda(x) = \sum_{i=1}^{n} \lambda_i u_i(x_i)$, takes its maximum at a feasible allocation $x(\lambda)$ such that $u(x(\lambda)) = u \in UP$. If utilities are strictly concave, then this solution is unique.

There exists a reciprocal for this proposition.

Proposition 3. Given a Pareto optimal allocation $\bar{x}$, *t*here exists a vector $\bar{\lambda} \in \Delta^{n-1}$ such that $\bar{x}$ solves the maximization problem:

$$max_{x \in F} \sum_{i=1}^{n} \bar{\lambda}_i u_i(x)$$

i.e., $U_{\bar{\lambda}}(\bar{x}) \geq U_{\bar{\lambda}}(x) \ \forall \ x \in F$.

Proof: Notice that if the allocation $\bar{x}$ is Pareto Optimal then $\bar{u} = u(\bar{x})$ is in the boundary of the utility possibility set. Since this is a convex set, by the supporting hyperplane theorem, there exists $\bar{\lambda} \ngeq 0$ such that $\bar{\lambda}\bar{u} \geq \bar{\lambda}u \ \forall \ u \in UP$.

In principle, utilitarian welfare comparisons could be made in a different manner by different choice of the cardinal utilities, nevertheless the characterization of the eficient allocation is complete independent of this choice.

In the next sections we will analyze the relationship between efficiency, stability and egalitarianism.

3. Efficiency and egalitarianism

In the context of assessing the relative social desirability of alternative feasible allocations, the concept of Pareto efficiency is the central cornerstone of normative economics. However, some Pareto optimal allocations may be inequitable from some distributional point of view. One would like to supplement the Pareto condition with some notion of economic justice. With the purpose, one of the most studied solution for a bargaining problem, is the egalitarian solution. This solution was recommended in J. Rawls (1999). Given a bargaining problem faced by individuals in the society, this solution implies maximization of the utility of the worst off individual over the bargaining set. A characterization of the egalitarian solution when the number of individuals is fixed and the bargaining set is convex, compact, and comprehensive was proposed in Kalai (1977) using symmetry and weak Pareto optimality.

As we have shown in the previous section, given a vector $\lambda \in \Delta^{n-1}$ there exists a Pareto optimal allocation $x^*(\lambda)$ such that:

$$U_\lambda\big(x^*(\lambda)\big) \geq U_\lambda(x)\ \forall x \in \mathcal{F} \tag{6}$$

Let us introduce the function $\widetilde{U}: \Delta^{n-1} \to R$ defined by:

$$\widetilde{U}(\lambda) = U(\lambda, x^*(\lambda)) = \sum_{i=1}^{n} \lambda_i u_i(x_i^*(\lambda))$$

where $x(\lambda)$ is the Pareto optimal allocation such that $u \in UP,$ verifying $u = u(x^*(\lambda)))$ and being $\lambda = \xi(u)$.

Now we introduce some considerations on the egalitarian allocation, x^e, where by egalitarian we mean, a Pareto optimal allocation such that every individual attains the same level of utility.

Assume that each utility function is Gateaux-differentiable for every $x_i \in E^+$ in every admissible direction, verifying that for each $i \in \{1, \dots, n\}$ there exists a continuous linear operator, $u'_i(x_i) \in L(E, R)$ such that:

$$u_i(x_i + tv) - u_i(x_i) = tu'_i v + o(t),$$

for all $v \in A_{x_i}$ and $t \in (0, \delta)$, for some $\delta > 0$.

We also assume that for each utility function, every $x_i \in E^+$ and admissible direction k, there exists a derivative of u'_i i.e., $u''_i(x_i)$ such that:

$$u'_i(x_i + tv) - u'_i(x) = tu_{i''}(x_i)v + o(t) \quad \forall i \in \{1, \dots, n\}.$$

Then, $u''_i(x_i)$ is a continuous linear operator from E to $L(E, R)$, i.e.,

$$u''_i(x_i) \in L(E, L(E, R)).$$

If we choose $h, k \in E$ then $u''_i(x_i)h \in L(E, R)$ and $u''_i(x_i)hk \in R$.

Fixing $\lambda = \lambda^s$, the first order conditions for de maximization problem (5) corresponding to a solution x^s are given by:

$$\lambda_i^s u'_i(x_i^s) = (\gamma^s)^*(I), \quad i = \{1, \dots, n\}, \tag{7}$$

where $(\gamma^s)^* \in E^*$ is the Lagrange's multiplier corresponding to the restriction $g(x) = \sum_{i=1}^n x_i - W = 0$, and I is the identity function in E, by E^* we symbolize the dual space of E. See appendix.

Let us consider the function $F: \Delta \times E^m \times E^* \to R^n$ defined as:

$$F_i(\lambda, x, \gamma) = \lambda_i u'_i(x_i) - \gamma, \quad i = \{1, \dots, n\},$$

where $\gamma \in E^*$. Since $(F'_x, F'_\gamma): E^n \times E^* \to R^n$ is surjective, then from the surjective implicit function theorem (see Zeidler, 1993), it is possible to find a suitable neighborhood $V_{\lambda^s} \subset R^n \cap \Delta$, of λ^s, and C^1 functions $x: V_{\lambda^s} \to (E^+)^m$ and $\gamma: V_{\lambda^s} \to E^*$ such that $F(\lambda, x(\lambda), \gamma(\lambda)) = 0 \;\forall\; \lambda \in V_{\lambda^s}$ verifying $x(\lambda^s) = x^s$ and $\gamma(\lambda^s) = \gamma^s$. The surjective implicit function theorem can be extended to the application $A: U \times V \to Z$, where U is a convex subset of a Banach manifolds M, V is an open subset in a Banach manifold Q and Z is a Banach space (see Accinelli, 2010).

Then the following proposition holds.

Proposition 4. The egalitarian allocation x^e solving $u_i(x_i^e) = u^e, \;\forall i \in I$ is the Pareto optimal allocation corresponding to the solution of the minimization problem:

$$\min_{\lambda \in \Delta^{n-1}} U(\lambda, x^*(\lambda)) = \sum_{i=1}^{n} \lambda_i u_i(x_i^*(\lambda))$$

Proof: In Accinelli et al. (2008) is shown that the function $\widetilde{U}(\lambda) = U(\lambda, x(\lambda))$ is strictly convex. So, the first order condition is a necessary and sufficient condition for minimization. Let λ^e be the solution of this problem. It follows that

$$u_i(x_i^*(\lambda^e)) = u_j(x_j^*(\lambda^e)) = U(\lambda^e, x^*(\lambda^e) = \widetilde{U}(\lambda^e) \;\forall\; i, j \in I.$$

Note that, under the main hypphotesis of this work the egalitarian allocation always exists.

The existence is a direct consequence of the convexity of the function $\widetilde{U}(\lambda) = (U\big(\lambda, x^*(\lambda)\big)$. And it is the only efficient allocation where every consumer attains the same level of utility. Its existence does not depend on the utilities representing the preferences. Moreover, if the same monotone increasing transformation is applied to every utility function, this solution remains the same. Such allocation is efficient from the Pareto point of view, and supplemented with the additional property that every agent enjoy precisely the same level of welfare, however no necessarily each one obtains the same bundle. The concept of fairness that this egalitarian allocation introduces does not mean free envy allocation in strict sense, but it guarantee the same level of happiness for all the agents of the economy.

It is important to observe that the existence and the efficiency of the egalitarian allocation is independent of the preference representation.

Proposition 5. Let x^e be the egalitarian Pareto optimal allocation and let $\lambda^e \in \Delta$ such that x^e solves the problem

$$\max_{x \in F} \sum_{i=1}^{n} \lambda_i^e u_i(x)$$

then $\lambda^e = (\frac{1}{n}, \dots \frac{1}{n})$.

Proof: Let $u^e = (u_1^e, \dots, u_n^e)$ be the utility corresponding to the egalitarian allocation x^e. Suppose that $\lambda^e = (\lambda_1, \dots, \lambda_n) \in \Delta$ such that $\lambda_h > \lambda_i \ \forall \ i \neq h$. Now consider the feasible allocation $x_\alpha = (x_1^e - \frac{\alpha}{n-1}, \dots, x_h^e + \alpha, \dots., x_n^e - \frac{\alpha}{n-1})$ where $\alpha \in R_+^n$ and the corresponding utility vector $u(x_\alpha) = u_\varepsilon$. Since utilities are monotones then there exist some $\varepsilon > 0$ such that $u_\varepsilon = (u_1^e - \frac{\varepsilon}{m-1}, \dots, u_h^e + \varepsilon, \dots., u_n^e - \frac{\varepsilon}{m-1})$, It follows that $\lambda^e u_\varepsilon > \lambda^e u^e$. this is not possible because of the assumption that x^e maximize $U_{\lambda^e}(x), \forall x \in F$.

In principle, utilitarian welfare comparisons could be made in a different manner if we choose different sets of utility functions representing the same preferences. Specifically, it is possible to obtain different conclusions over the welfare levels attained in a particular allocation. However, the main characteristics of the egalitarian allocation (fairness and efficiency) are independent of the utility functions representing the preferences of the agents. Moreover, fixing the economy E and assuming strictly convex preferences, this allocation is unique.

Definition 4. We say that an economy $E = \{X_i, u_i, w_i, I\}$ is neoclassical, if the consumption set X_i is the positive cone of a Banach lattice E (the same for every consumer), the endowments are strictly positive, the utilities are strictly concave and twice Gateaux-differentiable in every $x \in E^+$ and in every admissible direction.

Proposition 6. Given a neoclassical economy $E = \{X_i, u_i, w_i, I\}$, the average value of the utilities achieved in any optimal allocation, is less than the value of the social utility corresponding to the egalitarian allocation, i.e.;

$$\sum_{i=1}^{n} \frac{1}{n} u_i(x) \leq \sum_{i=1}^{n} \lambda^e u_i(x(\lambda^e)).$$

Proof: Let $x = (x_1, \dots, x_n)$ be a Pareto optimal allocation, and consider $\overline{U}(x) = \sum_{i=1}^{n} \frac{1}{n} u_i(x)$. The prove is straightforward from de definition. Certainly if $\overline{U}(x) > U_{\lambda^e}(x^e)$, then x^e is not the Pareto optimal allocation maximizing $U_{\lambda^e}(x)\ \forall x \in PO$.

Since $\lambda^e = 1/n$ proposition states that the average level of satisfaction achieved by society is maximum for egalitarian allocation.

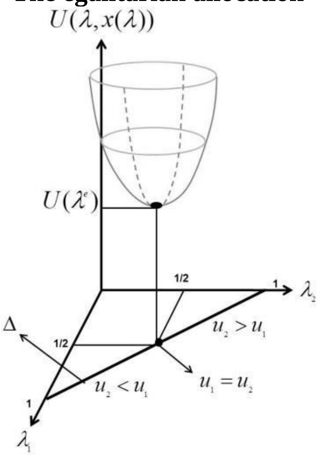
4. Stability of the egalitarian solution

The concept of stability of the egalitarian allocation that we introduce here, is closely related with the axiomatic characterization given in Thomsom (1984). Among all efficient allocations, the egalitarian is the only one such that every people is equally happy, and consequently, verifying that the representative agent obtains the same level of utility every consumer in the economy. More precisely, this allocation ensures *"equal treatment of equals"*. Note that if the social weight $\lambda_i = 1/n$ is the same for every individual in the economy, then the egalitarian allocation is the only one Pareto optimal allocation solving the maximization of the social utility function $U_{(\frac{1}{n},\dots,\frac{1}{n})}(x)\ \forall\ x \in F$. Moreover, it is the only one that can be considered as equitable for any subgroup, and is, precisely, such characteristic that gives to this allocation a certain degree of stability. On the other hand, according with proposition (6), this allocation ensures the maximum (per capita) level of welfare, that can be achieved in an efficient way in a given economy.

Paradoxically, it is because of the inequality is an attribute of most economies and of any complex society, which makes it interesting to consider the egalitarian allocation. It becomes a benchmark from which the degree of inequality that each allocation involves can be measured. This property, together with the relative degree of justice it ensures, are strong incentives to consider such allocation.

Note that any change in the parameters of an egalitarian economy, imply that necessarily one agent, at least, attains a higher level of utility but in detriment of the rest of the society, (see figure (3). So, after any perturbation in the fundamentals of the egalitarian economy, the rest of the society will push to return to the egalitarian situation. In this sense it is possible to say that the egalitarian solution corresponds to an efficient and consensual wealth distribution. Is worthless to note that the egalitarian allocation imply equal level of happiness, but not necessarily the same consumption bundle for every one.

Figure 3
The egalitarian allocation



The egalitarian distribution can be attained in a decentralized way, if and only if the distribution of the initial endowments allow that this allocation can be attained as the result of a Walrasian equilibrium.

If the egalitarian allocation is reached as a Walrasian allocation, then the social weights of all consumers are the same, the intuition behind this affirmation is that under this equilibrium, the different social groups or consumers in the economy, have similar economic power. So, this situation implies that: "An economy remains in force so long as no party wishes to defect to the noncooperative situation, and it is reinstituted as soon as each party finds it to its advantage to revert to cooperation" (see Barbosa, Jovanovic and Spiegel, 1997).

Let us define the function $\psi: UP \rightarrow \Delta^{n-1}$ given by $\psi(\bar{u}) = \bar{\lambda}$ such that $\bar{\lambda}\bar{u} \geq \bar{\lambda}u, \forall u \in U$. The following definition is equivalent to the definition of the Negishi path given in Accinelli, Hernández and Plata (2008).

Definition 5. The path $NPU = \{(\lambda, \psi^{-1}(\lambda)), \lambda \in \Delta^{n-1}\}$ will be called the Negishi utility path.

Along the Negishi path we find the set of pairs $(\lambda, u^{\lambda}) \in \Delta^{n-1} \times \mathcal{UP}$ such that the levels of individuals welfare defined by u^{λ} correspond to efficient allocations, and the corresponding distribution of social weights.

Consider the function $\widetilde{U}: \mathcal{NPU} \to R$ defined by

$$\widetilde{U}(\lambda, \psi^{-1}(\lambda)) = \sum_{i=1}^{n} \lambda_i u_i^{\lambda}$$

where $\psi^{-1}(\lambda) = u^{\lambda} = (u_1^{\lambda}, \ldots, u_n^{\lambda})$. This function, defined along the Negishi utility path reaches its minimum at λ^e i.e: $\forall\ \lambda \in \Delta^{n-1}$:

$$\widetilde{U}((\lambda, \psi^{-1}(\lambda)) = \widetilde{U}(\lambda, u^{\lambda}) \geq \widetilde{U}(\lambda^e, u^e) = \widetilde{U}(\lambda^e, \psi^{-1}(\lambda^e)).$$

In John Rawls's theory of justice, it is asserted that institutions and practices should be arranged so that the worst off are as well off over the long run as possible, they work to the maximal advantage of the worst off members of society, (see Rawls, 1999 and 2001).

Precisely, the utility obtained from the egalitarian allocation corresponds to the solution of maximizing the utility of those individuals who achieve worse results, i.e.,

$$u^e = max_{u \in \mathcal{UP}} \{min\{u_1, \ldots u_n\}\}$$

Thus the distribution of resources that allows to reach justice in the Rawslian sense, also ensures social stability, and corresponds to the cooperative solution, even if this is achieved in a decentralized way.

Several works (see for instance Bowles and Herbert, 1998) show that the fact that more equal countries have more rapid rates of economic growth, could be well accounted for a statistical association between measures of equality and unmeasured causes of economic growth. This observation does not imply, that equality “per se” promotes high levels of economic performance, but egalitarian policies are compatible with the rapid growth of productivity. The capitalist countries taken as a whole have grown faster under the aegis of the post Second World War than in any other period, and in this was the period of ascendent welfare state and social democracy, both of them promoting the egalitarianism between the different agents of the society.

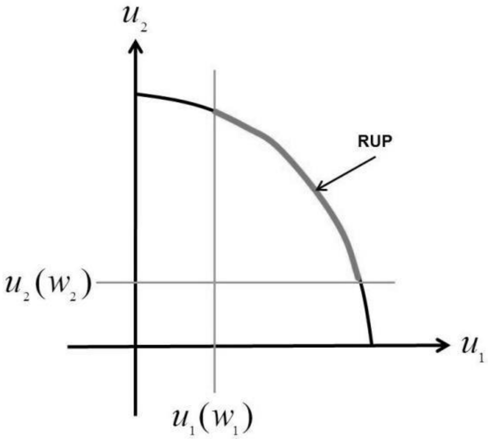
5. Efficiency, egalitarianism and markets

In this section we discuss the possibility that an egalitarian distribution of the social resources can be the result of the only action of the markets law. The standard model of general equilibrium shows that the equilibrium allocations are Pareto efficient, however, the relation between equilibria and fairness is, until now, object of discussing.

The main properties of egalitarian allocations are: 1) The utility functions assume the same value for everyone, 2) they are Pareto optimal allocations and 3) to all agents, correspond the same social weight. This means that, these allocations can be implemented as an equilibrium allocation. Under this equilibrium every agent will have the same weight in the market. This means that every agent is equal, and the egalitarian allocation ensures that equals receive the same treatment. The main question of this section is if an economy based on free markets can attain the egalitarian allocation in a decentralized way.

The agents go to the market with the purpose of finding a bundle set preferable to their endowments, i.e., the $i-th$ agent go to the market to find a bundle set $x_i \in E_+ : u_i(x_i) \geq u_i(w_i),\ i = 1,2,\ldots,n$. Only an allocation being part of a Walrasian equilibrium can be attained in a decentralized way. From the first welfare theorem such allocations $x \in E^n$ are Pareto optimal, and given the rationality of the agents, these allocation must verify that $u_i(x_i) \geq u_i(w_i), i = 1,\ldots,n$. We denote by $\mathcal{RPO}$ the set of allocations $x \in OP$ such that $u_i(x_i) \geq u_i(w_i),\ \forall\ i = 1,\ldots,n$. The corresponding levels of utility for these allocations are given by: $\mathcal{RUP} = \{u \in \mathcal{UP} : u_i \geq ui(xi)\ \forall i=1,2,\ldots,n$ see figure (4).

Figure 4
Rational Pareto optimal allocations



As it is well known, given an economy $\mathcal{E}$ a feasible allocation x^w is Walrasian if there exists a set of prices $p \in E^*$ such that the pair (x^w, p) is a Walrasian equilibrium for the economy $\mathcal{E}$. We will symbolize by $\mathcal{W}_{\mathcal{E}}$ the set of Walrasian allocations of a given economy $\mathcal{E}$.

The first welfare theorem establishes a relationship between Walrasian allocations and Pareto optimal allocations. Since the only of these Pareto optimal allocations can be achieved in a decentralized way, i.e., by the unique action of the laws of economics, are the Walrasian allocations, the possible levels of utilities attainable in a given economy, depend on the distribution of initial endowments. So, it is possible that for a given economy, with a very unequal distribution of the initial endowments can not be attained by the only action of the markets.

The second welfare theorem for economies with infinitely many commodities says that, if in a exchange economy, preferences are monotone, convex and uniformly τ proper then for any Pareto optimal allocation x^P, there exists a non zero price p such, that the pair (p, x^P) is an equilibrium with transfer payments $t_i = p(x_i^P - w_i)$[1] (for details see Mas Colell, 1986). In other words, under the above conditions (which are usually considered by the theory), a benevolent central planner can obtain, after transfers, that the egalitarian allocation is reached in a decentralized way, i.e., under the exclusive action of market laws. The egalitarian allocation can be the result of a previous agreement about the distribution on the aggregate endowment

[1] As usual the evaluation $p(x)$ is dented by $px \ \forall x \in E, \ and \ p \in E^*$.

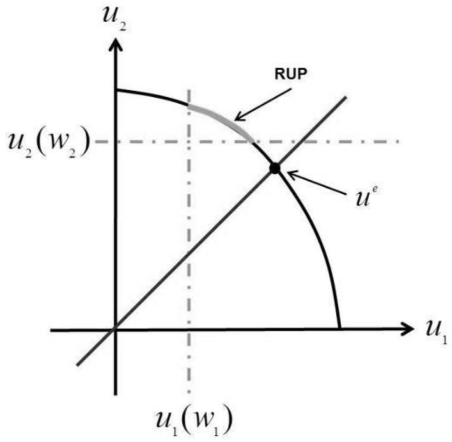
performed by the agents of a "quasi-rawlsian" economy. Unlike of the Rawlsian individuals who will choose an equal distribution of resources, our agents are choosing a Pareto efficient distributions, able to be attained in a decentralized way, by the free action of the markets. Note that the equal distribution of resources (the same for each agent), a difference of the egalitarian allocation, is not necessarily a Pareto optimal allocation.

Let us define an unequal economy:

Definition 6. An economy E is unequal if the egalitarian allocation x^e is not an individually rational Pareto optimal allocation. That is, $x^e \not\in RPO$.

This situation corresponds to an economy E where the initial distribution of the resources is very unevenly So, a unequal economy, whose agents are rational, can not attain an egalitarian distribution of wealth by the only action of the markets (see figure 5). To attain certain degree of social justice, starting with an excessively unequal distribution of the initial endowments, implies the participation of a central planer able to implement a set of economic policy measures to this end. This affirmation can be summarized in the next proposition:

Figure 5
An unequal economy



Proposition 7. Given an unequal economy, the egalitarian distribution x^e can not be attained in a decentralized way.

Proof: Since $x^e \not\in \mathcal{RPO}$ there is a neighborhood $V_{x^e} \subset E^{ln}$ of this allocation such that no allocation in $UP \cap V_{x^e}$ can be a Walrasian allocation.•

Corolary 1. In an unequal economy, there exists $\varepsilon > 0$ such that the levels of utility u^w corresponding with a Walrasian allocation verify the inequality*:* $|u^w - u^e| > \varepsilon$.

Corolary 2. In an unequal economy, $u^e \not\subseteq RUP$

Let $x^w \in \mathcal{W}_\varepsilon$ be a walrasian allocation, the ratio $\frac{u_i(x_i^w)}{u_i(w_i)}$ measures the relative value that the $i-th$ consumer assigns to the market allocation, and the ratio $\frac{u_i(x_i^e)}{u_i(w_i)}$ measures the relative value that the $i-th$ consumer assigns to the egalitarian allocation. A consumer prefers the Walrasian allocation x^w to the egalitarian allocation x^e if and only if $\frac{u_i^w}{u_i(w_i)} > \frac{u_i^e}{u_i(w_i)}$ where $u_i^w = u_i(x_i^w)$ and $u_i^e = u_i(x_i^e)$.

Let us define the subset $\mathcal{U}^w \subset \mathcal{U}$ where

$$\mathcal{U}^w = \{u^w \in R^n \ : \ there\ \ exists\ \ x^w \in \mathcal{W}_\varepsilon\ \ such\ \ that\ \ u^w = u(x^w)\}$$

This subset captures the attainable vectors of the utility values that can be obtained by means of a Walrasian allocation.

One of the most important challenges for modern economic theory consists in measuring inequality. With the purpose to contribute in this sense we introduce de following two indexes.

Definition 7. The following index measures how far a given economy E_{un} is to achieve in a decentralized way an equal distribution:

$$I_E = min_{u^w \in U^w} \sum_{i}^{n} |u_i(x_i^w) - u_i(x_i^e)|.$$

If for a given economy, this index is positive, then the equal distribution can be achieved only after transfers. Since utilities are not observable we can measure the degree of inequality of an economy from de following index:

Definition 8. The following index measures how far a given economy E_{un} is to achieve in a decentralized way an equal distribution:

$$J_E = min_{x^w \in W_E} \sum_{i=1}^{n} \| x_i^w - x_i^e \|.$$

Since the utilities are Gateaux-differentiable it follows that

$$|u_i(x_i) - u_i(x_i^e)| = |u'_i(x_i^e)| \| x_i - x_i^e \| + o(\| x_i - x_i^e \|)$$

and given that from the first order condition for the maximization problem (5), the value $\lambda_i^e u'_i(x_i^e) = \lambda_i^e u'$ is the same for all agents and $\lambda_i^e = 1/n,\ i = 1,\dots,n$ (see proposition (5)); then, in a neighborhood of the allocation x^e, the indices in definitions (7) and (8) verify the relation

$$I_E \simeq u' J_E.$$

The following proposition characterizes an unequal economy:

Proposition 8. Let E be an economy which endowments are $W = (w_1,\dots,w_n)$. The economy is unequal if and only if there exists an individual i such that $u_i(w_i) > u_i(x_i^e) = \widetilde{U}(\lambda^e, x(\lambda^e))$.

Proof: Since every Walrasian allocation x^w economy, must verify that $u_i(x_i^w) \geq u_i(w_i)$ and since in this case, $u_i(w_i) > u_i^e$, then the egalitarian allocation x^e can not be a Walrasian allocation for $\mathcal{E}$.

This proposition is shown in figure (5). Note that the definition of unequal economy does not depend on the utilities representing the preferences of the consumers.

In accordance with propositions (7) and (8) economies with a large number of individuals under the poverty line or with unequal opportunity set across the individuals, the only action of the markets, probably gives place to economies with high indices of inequality. However the second welfare theorem says that after transfers, it is possible to obtain a functional $p \in E^*$ supporting the egalitarian allocation as a Walrasian allocation. So, to obtain an egalitarian economy starting from an unequal economy it is necessary to implement a set of measures of political economy. Recall that an element

$p \in E^*$ supports the allocation x if for each allocation y such that $u_i(y_i) > u_i(x_i)$ then, $p(y_i) > p(x_i)$ $\forall$ $i \in I$. Then the pair (x, p) is a walrasian equilibrium with transfers.

Notice that at the same time that an economy approachs the egalitarian solution, the social weights of the different agents tend to be equal.

6. An example

Consider an exchange economy with R^2 as consumption space having two consumers with initial endowments $w^1 = (w_1^1, w_2^1)$ and $w^2 = (w_1^2, w_2^2)$, and preferences represented by the utility functions: $u_1(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$, and $u_2(x_1, x_2) = x_1 x_2$.

Let $x = (x_1^1, x_2^1; x_1^2, x_2^2)$ be an allocation. The social welfare function is:

$$U_\lambda(x) = \lambda_1 (x_1^1)^{\frac{1}{2}} (x_2^1)^{\frac{1}{2}} + \lambda_2 x_1^2 x_2^2. \tag{8}$$

To obtain the Pareto optimal allocations, we need to solve the maximization problem

$$\max_{x \in R_+^2 \times R_+^2} U_\lambda(x) = \lambda_1 (x_1^1)^{\frac{1}{2}} (x_2^1)^{\frac{1}{2}} + \lambda_2 x_1 x_2,$$

$$s.t, x_1^1 + x_1^2 = w_1^1 + w_1^2, \tag{9}$$

$$x_2^1 + x_2^2 = w_2^1 + w_2^2.$$

To solve such problem we consider

$$U(\lambda) = \lambda_1 (x_1^1)^{\frac{1}{2}} (x_2^1)^{\frac{1}{2}} + \lambda_2 (W_1 - x_1)(W_2 x_2) \tag{10}$$

where $W_1 = w_1^1 + w_1^2$ and $W_2 = w_2^1 + w_2^2$.

Taking derivatives we obtain:

$$\frac{1}{2}\lambda_1 x_1^{-\frac{1}{2}} x_2^{\frac{1}{2}} - \lambda_2[W_2 - x_2] = 0$$

$$\frac{1}{2}\lambda_1 x_1^{\frac{1}{2}} x_2^{-\frac{1}{2}} - \lambda_2[W_1 - x_1] = 0$$

After some algebra we get:

$$x_1(\lambda_1,\lambda_2) = \frac{1}{2}\frac{\lambda_1}{\lambda_2}\left[\frac{w_1}{w_2}\right]^{\frac{1}{2}} - w_1$$
$$x_2(\lambda_1,\lambda_2) = \frac{1}{2}\frac{\lambda_1}{\lambda_2}\left[\frac{w_2}{w_1}\right]^{\frac{1}{2}} - w_2 \qquad (11)$$

The egalitarian solution correspond to $\lambda_1 = \lambda_2 = \frac{1}{2}$, and it is

$$x_1^e = \frac{1}{2}\left[\frac{w_1}{w_2}\right]^{\frac{1}{2}} - w_1$$
$$x_2^e = \frac{1}{2}\left[\frac{w_2}{w_1}\right]^{\frac{1}{2}} - w_2 \qquad (12)$$

Following the Negishi approach, an allocation is a Walrasian allocation if and only if the equations

$$e_1(\lambda_1,\lambda_2) = 0$$
$$e_2(\lambda_1,\lambda_2) = 0$$

are verified. Since $<\lambda, e(\lambda)> = 0 \quad \forall \quad \lambda \in \Delta$, it is enough to solve one of the last two equations to characterize the equilibria allocations. So, we choose the first one and we solve:

$$e_1(\lambda_1,\lambda_2) = \lambda_1\frac{\partial u_1}{\partial x_1}[x_1^1(\lambda_1,\lambda_2) - w_1^1] + \lambda_2\frac{\partial u_1}{\partial x_2}[x_2^1(\lambda_1,\lambda_2) - w_2^1] = 0$$

After some algebra it follows that a solution of this equation verifies the relation:

$$\lambda_1 x_2^1 w_1^1 = \lambda_2 w_2^1 x_1^1$$

Substituting in equation (11) we obtain that

$$\lambda = (\lambda_1, \lambda_2) \in \Delta$$

solves $e(\lambda) = 0$ if and only if:

$$\lambda_1 \left[\frac{1}{2}\frac{\lambda_1}{\lambda_2}\left[\frac{w_2}{w_1}\right]^{\frac{1}{2}} - w_2\right] w_1^1 = \lambda_2 \left[\frac{1}{2}\frac{\lambda_1}{\lambda_2}\left[\frac{w_1}{w_2}\right]^{\frac{1}{2}} - w_1\right] w_1^2$$

It follows that, the egalitarian allocation is an equilibrium allocation if and only if:

$$\frac{w_1^1}{w_2^1} = \frac{[(\frac{w_1}{w_2})^{\frac{1}{2}} - w_2]}{[(\frac{w_2}{w_1})^{\frac{1}{2}} - w_1]}$$

Conclusion

Free markets ensure efficiency but in some cases they can not ensure an egalitarian allocation. In some cases the only possible Walrasian allocations to be reached by the only action of the free markets, have associate a very unequal levels of happiness. Obviously, this situation give place to a very unstable society, where more unhappy people can recruit for potential violent movements. We introduce an index to measure the level of inequality of a given economy. To the best of our knowledge, this is new in the literature.

In these cases the participation of a central planner can introduce stability in the economy, if he is able to implement measures diminishing inequality. However, as is increasingly recognized, the intervention of a central authority to alter the distribution of the income can be accompanied of heavy political and economic costs. On the other hand, those who would be harmed by these policies (the wealthy) can organize effective political opposition.

An alternative policy able to alter the distribution of wealth may be to encourage investment in technology and human capital increasing in this way the endowments of the workers. Technologically developed firms will be able to get more productivity and also pay higher wages to their workers, in particular for skilled workers (see Accienlli and Carrera, 2012).

Finally, a comment on the hyphotesis of strict concavity of the utility functions. It is possible to extend the method considering only concavity, however although this would give more generality, the loss of uniqueness of the solution of the maximization problem of proposition 3, would require us to define some criteria to choose the efficient allocation corresponding to a given distribution of social weights. This task will subject of future works.

Appendix

The Lagrange multiplier theorem for Banach spaces

Let X and Y be real Banach spaces. Let $f: X^+ \to R$ a twice Gateaux-differentiable function in every admissible direction. Let $g: X^+ \to Y$ be another twice Gateaux-differentiable function in every admissible direction, the constraint: the objective is to find the extremal points (maxima or minima) of f subject to the constraint that g is zero.

Suppose that $x_0 \in X^+$ is a constrained extremum of f in X^+ i.e. an extremum of f on

$$g^{-1}(0) = \{x \in X^+ : g(x) = 0 \in Y\}.$$

Suppose also that the Gateaux derivative in every admissible direction is defined by $g'(x_0): X \to Y$ an it is a surjective linear map. Then there exists a Lagrange multiplier $\gamma: Y \to R$ in Y^*, the dual space to Y, such that $f'x_0) = \gamma \circ g'(x_0)$. See Luemberger (1969).

References

Accinelli, E. (2010). “A Generalization of the Implicit Function Theorem”. *Applied Mathematical Sciences*, 4(26), 1289-1298.

Accinelli, E., Brida, G., Plata, L., and Puchet. M. (2008). "Bienestar social, óptimos de Pareto y equilibrios walrasianos". *El Trimestre Económico*, LXXV(número especial para las VII Jolate).

Accienlli, E., and Carrera, E. (2012). "The Evolutionary Game of Poverty Traps". *The Manchester School, Special Issue on Poverty Traps* Vol. (80)/4, pp.381-400.

Accinelli, E., and Plata, L. (2011). Microeconomic Foundations of the Social. In *Dynamic, Game and Science: in Honour of Mauricio Peixoto and David Rand* (42-54). Series: Springer-Veralg.

Accinelli, E., Hernández, R., and Plata, L. (2008). "Eficiencia y bienestar social: El camino de Negishi". *Revista Perspectivas* (UASLP), 3(4), 1-20.

Aliprantis, C. D., Brown, D. J., and Burkinshaw, O. (1990). *"Existence and Optimality of Competitive Equilibrium".* Springer-Verlag.

Arrow, K. (1951). *Social Choice and Individual Values*. New York: J. Willey and Sons.

Barbosa, P., Jovanovic, B., and Spiegel, M. (1997). "Inequality and Stability". *Annales d' Economie et de Statistique*, 48, 15-40.

Bowles, S., and Herbert, G. (1998). *Recasting egalitarianism: new rules for communities, states and markets*. Verso.

d'Aspremont, C., and Gevers, L. (1977). "Equity and the informational basis of colleclive choice". *Review of Eiconomic Studies*, 199-203.

Hammond , P. J. (1976). "Equity, Arrow's conditions and Rawls' difference principle". *Econometrica*, 44, 793-804.

Imair, H. (1983). "Individual monotonicity and lexicographic maxima solutions". Econometrica 51 pp. 389-401.

Kalai, E. (1977). "Proportional solutions to bargaining situations: Interpersonal utility comparisons". *Econometrica*, 45, 1623-1630.

Luemberger D. (1969). "Optimization by Vector Space Methods". Ed, by John Wiley and Sons.

Mas Colell, A. (1986). "The price existence problem in topotlogical lattices". *Econometrica*, 54, 1039-1053.

Nash, J. F. (1950). "The bargaining problem". *Econometrica*, 8, 155-162.

Negishi, T. (1960). "Welfare economics and existence of an equilibrium for a competitive economy". *Metroeconomics*, 12, 92-97

Rawls, J. (1999). *A Theory of Justice*. Section 12 revised edition, Cambridge: Harvard University Press.

______ (2001). Justice as Fairness: A Restatement. Section 13, Ed. by Erin Kelly, Cambridge: Harvard University Press.

Roth, A. (1979). "Proportional1 solutions to the bargaining problem". *Econometrica*, 47, 775-778.

Sen, A. K. (1977). "On weights and measures: Informatinal constriants in social welfare analysis". *Econometrica*, 45, 1539-1572.

Strasnick, S. (1976). "Social choice and the derivation of Rawls' difference principle". *The Journal of Philosophy*, 73, 85-99.

Thomsom, W. (1984). "Monotonicity, Stability and Egalitarianism". *Mathematical Social Sciences*, 8, 15-28.

Zeidler, E. (1993). *Nonlinear Functional Analysis and its Applications*. Vol I, Ed. Springer-Verlag.